

Perturbation theory methods applied to critical phenomena

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Abstract

Different perturbation theory treatments of the Ginzburg–Landau phase transition model are discussed. This includes a criticism of the perturbative renormalization group (RG) approach and a proposal of a novel method providing critical exponents consistent with the known exact solutions in two dimensions. The new values of critical exponents are discussed and compared to the results of numerical simulations and experiments.

Keywords: Ginzburg–Landau model, Feynman diagrams, renormalization group, critical exponents, quenched randomness.

1 Introduction

Phase transitions and critical phenomena is one of the most widely investigated topics in modern physics. Nevertheless, a limited number of exact and rigorous results is available [1]. Our purpose is to give a critical analysis of the conventional approach in calculation of critical exponents based on the perturbative renormalization group (RG) theory [2, 3, 4] and to propose a new method which provides results consistent with the known exact solutions. The basic hypothesis of the conventional (RG) theory is the existence of a certain fixed point for the RG transformation. However, the existence of such a stable fixed point for the Ginzburg–Landau model (which lies in the basis of the field theory) has not been proven mathematically in the case of the spatial dimensionality $d < 4$.

The usual RG theory treatment of the Ginzburg–Landau model is based on the diagrammatic perturbation theory (Feynman diagrams). We have demonstrated that this treatment is contradictory and therefore cannot give correct values of critical exponents. Namely, based on a method which is mathematically correct and well justified in view of the conventional RG theory, we prove the nonexistence of the non–Gaussian fixed point predicted by this theory (Sect. 2). In Sect. 3 we prove that a correctly treated diagram

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expansion provides results which essentially differ from those of the perturbative (diagrammatic) RG theory. Finally, we have proposed a novel analytical method of determination of critical exponents in the Ginzburg–Landau model (Sect. 4), and have compared the predicted exact values of critical exponents to the results of numerical and real experiments (Sect. 5).

2 Critical analysis of the perturbative RG method

Here we consider the Ginzburg–Landau phase transition model within the usual renormalization group approach to show that this approach is contradictory (for more details see also [10]). The Hamiltonian of this model in the Fourier representation reads

$$\frac{H}{T} = \sum_{\mathbf{k}} \left(r_0 + c \mathbf{k}^2 \right) |\varphi_{\mathbf{k}}|^2 + u V^{-1} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3} \varphi_{\mathbf{k}_1} \varphi_{\mathbf{k}_2} \varphi_{\mathbf{k}_3} \varphi_{-\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3} , \quad (1)$$

where $\varphi_{\mathbf{k}} = V^{-1/2} \int \varphi(\mathbf{x}) \exp(-i\mathbf{k}\mathbf{x}) d\mathbf{x}$ are Fourier components of the scalar order parameter field $\varphi(\mathbf{x})$, T is the temperature, and V is the volume of the system. In the RG field theory [3, 4] Hamiltonian (1) is renormalized by integration of $\exp(-H/T)$ over $\varphi_{\mathbf{k}}$ with $\Lambda/s < k < \Lambda$, followed by a certain rescaling procedure providing a Hamiltonian corresponding to the initial values of V and Λ , where Λ is the upper cutoff of the φ^4 interaction. Due to this procedure, additional terms appear in the Hamiltonian (1), so that in general the renormalized Hamiltonian contains a continuum of parameters. The basic hypothesis of the RG theory in $d < 4$ dimensions is the existence of a non-Gaussian fixed point $\mu = \mu^*$ for the RG transformation R_s defined in the space of Hamiltonian parameters, i.e.,

$$R_s \mu^* = \mu^* . \quad (2)$$

The fixed-point values of the Hamiltonian parameters are marked by an asterisk (r_0^* , c^* , and u^* , in particular). Note that μ^* is unambiguously defined by fixing the values of c^* and Λ . According to the RG theory, the main terms in the renormalized Hamiltonian in $d = 4 - \epsilon$ dimensions are those contained in (1) with r_0^* and u^* of the order ϵ , whereas the additional terms are small corrections of order ϵ^2 .

Consider the Fourier transform $G(\mathbf{k}, \mu)$ of the two-point correlation (Green's) function, corresponding to a point μ . Under the RG transformation R_s this function transforms as follows [3]

$$G(\mathbf{k}, \mu) = s^{2-\eta} G(s\mathbf{k}, R_s \mu) . \quad (3)$$

Let $G(\mathbf{k}, \mu) \equiv G(k, \mu)$ (at $\mathbf{k} \neq \mathbf{0}$ and $V \rightarrow \infty$) be defined within $k \leq \Lambda$. Since Eq. (3) holds for any $s > 1$, we can set $s = \Lambda/k$, which at $\mu = \mu^*$ yields

$$G(\mathbf{k}, \mu^*) = a k^{-2+\eta} \quad \text{for } k < \Lambda , \quad (4)$$

where $a = \Lambda^{2-\eta} G(\Lambda, \mu^*)$ is the amplitude and η is the universal critical exponent. According to the universality hypothesis, the infrared behavior of the Green's function is described by the same universal value of η at any μ on the critical surface (with the only requirement that all parameters of Hamiltonian (1) are present), i.e.,

$$G(\mathbf{k}, \mu) = b(\mu) k^{-2+\eta} \quad \text{at } k \rightarrow 0 , \quad (5)$$

where

$$b(\mu) = \lim_{k \rightarrow 0} k^{2-\eta} G(\mathbf{k}, \mu) . \quad (6)$$

According to Eq. (3), which holds for any $s = s(k) > 1$ and for $s = \Lambda/k$ in particular, Eq. (6) reduces to

$$b(\mu) = \lim_{k \rightarrow 0} k^{2-\eta} s(k)^{2-\eta} G(s\mathbf{k}, R_s\mu) = a , \quad (7)$$

if the fixed point $\mu^* = \lim_{s \rightarrow \infty} R_s\mu$ exists. Let us define the function $X(\mathbf{k}, \mu)$ as $X(\mathbf{k}, \mu) = k^{-2}G^{-1}(\mathbf{k}, \mu)$. According to Eqs. (4), (5), and (7), we have (for $k < \Lambda$)

$$X(\mathbf{k}, \mu^*) = \frac{1}{a} k^{-\eta} \quad (8)$$

and

$$X(\mathbf{k}, \mu) = \frac{1}{a} k^{-\eta} + \delta X(\mathbf{k}, \mu) , \quad (9)$$

where μ belongs to the critical surface, and $\delta X(\mathbf{k}, \mu)$ denotes the correction-to-scaling term. From (8) and (9) we obtain the equation

$$\delta X(\mathbf{k}, \mu^* + \delta\mu) = X(\mathbf{k}, \mu^* + \delta\mu) - X(\mathbf{k}, \mu^*) , \quad (10)$$

where $\delta\mu = \mu - \mu^*$. Since Eq. (10) is true for any small deviation $\delta\mu$ satisfying the relation

$$\mu^* = \lim_{s \rightarrow \infty} R_s(\mu^* + \delta\mu) , \quad (11)$$

we choose $\delta\mu$ such that $\mu^* \Rightarrow \mu^* + \delta\mu$ corresponds to the variation of the Hamiltonian parameters $r_0^* \Rightarrow r_0^* + \delta r_0$, $c^* \Rightarrow c^* + \delta c$, and $u^* \Rightarrow u^* + \epsilon \times \Delta$, where Δ is a small constant. The values of δr_0 and δc are chosen to fit the critical surface and to meet the condition (11) at fixed $c^* = 1$ and $\Lambda = 1$. In particular, quantity δc is found $\delta c = B \epsilon^2 + o(\epsilon^3)$ with some (small) coefficient $B = B(\Delta)$, to compensate the shift in c of the order ϵ^2 due to the renormalization (cf. [3]). The formal ϵ -expansion of $\delta X(\mathbf{k}, \mu)$, defined by Eq. (10), can be obtained in the usual way from the perturbation theory. This yields

$$\delta X(\mathbf{k}, \mu) = \epsilon^2 [C_1(\Delta) + C_2(\Delta) \ln k] + o(\epsilon^3) \quad \text{at } k \rightarrow 0 , \quad (12)$$

where $C_1(\Delta)$ and $C_2(\Delta)$ ($C_2 \neq 0$) are coefficients independent on ϵ .

It is commonly accepted in the RG field theory to make an expansion like (12), obtained from the diagrammatic perturbation theory, to fit an asymptotic expansion in k powers, thus determining the critical exponents. In general, such a method is not rigorous since, obviously, there exist such functions which do not contribute to the asymptotic expansion in k powers at $k \rightarrow 0$, but give a contribution to the formal ϵ -expansion at any fixed k . Besides, the expansion coefficients do not vanish at $k \rightarrow 0$. Trivial examples of such functions are $\epsilon^m \exp(-\epsilon k^{-\epsilon})$ and $\epsilon^m [1 - \tanh(\epsilon k^{-\epsilon})]$ where m is integer. Nevertheless, according to the general ideas of the RG theory (not based on Eq. (10)), in the vicinity of the fixed point the asymptotic expansion

$$X(\mathbf{k}, \mu) = \frac{1}{a} k^{-\eta} + b_1 k^{\epsilon+o(\epsilon^2)} + b_2 k^{2+o(\epsilon)} + \dots \quad (13)$$

is valid not only at $k \rightarrow 0$, but within $k < \Lambda$. The latter means that terms of the kind $\epsilon^m \exp(-\epsilon k^{-\epsilon})$ are absent or negligible. Thus, if the fixed point does exist, then we can

obtain correct ϵ -expansion of $\delta X(\mathbf{k}, \mu)$ at small k by expanding the term $b_1 k^{\epsilon+o(\epsilon^2)}$ (with $b_1 = b_1(\epsilon, \Delta)$) in Eq. (13) in ϵ powers, and the result must agree with (12) at small Δ , at least. The latter, however, is impossible since Eq. (12) never agree with

$$\delta X(\mathbf{k}, \mu) = b_1(\epsilon, \Delta) \left[1 + \epsilon \ln k + o(\epsilon^2) \right] \quad (14)$$

obtained from (13) at $k \rightarrow 0$. Thus, in its very basics the perturbative RG method in $4 - \epsilon$ dimensions is contradictory. From this we can conclude that the initial assumption about existence of a certain fixed point, predicted by the RG field theory in $4 - \epsilon$ dimensions, is not valid.

3 A model with quenched randomness

Here we consider the Ginzburg–Landau phase transition model with $O(n)$ symmetry (i.e., the n -vector model) which includes a quenched randomness, i.e., a random temperature disorder (for more details see also [11]). One of the basic ideas of the perturbative RG theory is that n may be considered as a continuous parameter and the limit $n \rightarrow 0$ makes sense describing the self-avoiding random walk or statistics of polymers [3, 4]. We have proven rigorously that within the diagrammatic perturbation theory the quenched randomness does not change the critical exponents at $n \rightarrow 0$, which is in contrast to the prediction of the conventional RG theory formulated by means of the Feynman diagrams.

The Hamiltonian of the actually considered model is

$$\begin{aligned} H/T &= \int \left[(r_0 + \sqrt{u} f(\mathbf{x})) \varphi^2(\mathbf{x}) + c (\nabla \varphi(\mathbf{x}))^2 \right] d\mathbf{x} \\ &+ u V^{-1} \sum_{i,j,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_i(\mathbf{k}_1) \varphi_i(\mathbf{k}_2) u_{\mathbf{k}_1+\mathbf{k}_2} \varphi_j(\mathbf{k}_3) \varphi_j(-\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \end{aligned} \quad (15)$$

which includes a random temperature (or random mass) disorder represented by the term $\sqrt{u} f(\mathbf{x}) \varphi^2(\mathbf{x})$. For convenience, we call this model the random model. In Eq. (15) $\varphi(\mathbf{x})$ is an n -component vector with components $\varphi_i(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k} < \Lambda} \varphi_i(\mathbf{k}) e^{i\mathbf{k}\mathbf{x}}$, depending on the coordinate \mathbf{x} , and $f(\mathbf{x}) = V^{-1/2} \sum_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}$ is a random variable with the Fourier components $f_{\mathbf{k}} = V^{-1/2} \int f(\mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x}$.

The system is characterized by the two-point correlation function $G_i(\mathbf{k})$ defined by the equation

$$\langle \varphi_i(\mathbf{k}) \varphi_j(-\mathbf{k}) \rangle = \delta_{i,j} G_i(\mathbf{k}) = \delta_{i,j} G(\mathbf{k}) . \quad (16)$$

It is supposed that the averaging is performed over the $\varphi(\mathbf{x})$ configurations and then over the $f(\mathbf{x})$ configurations with a fixed (quenched) Gaussian distribution $P(\{f_{\mathbf{k}}\})$ for the set of Fourier components $\{f_{\mathbf{k}}\}$, i. e., our random model describes a quenched randomness.

We have proven the following theorem.

Theorem. In the limit $n \rightarrow 0$, the perturbation expansion of the correlation function $G(\mathbf{k})$ in u power series for the random model with the Hamiltonian (15) is identical to the perturbation expansion for the corresponding model with the Hamiltonian

$$\begin{aligned} H/T &= \int \left[r_0 \varphi^2(\mathbf{x}) + c (\nabla \varphi(\mathbf{x}))^2 \right] d\mathbf{x} \\ &+ u V^{-1} \sum_{i,j,\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \varphi_i(\mathbf{k}_1) \varphi_i(\mathbf{k}_2) \tilde{u}_{\mathbf{k}_1+\mathbf{k}_2} \varphi_j(\mathbf{k}_3) \varphi_j(-\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) \end{aligned} \quad (17)$$

where $\tilde{u}_{\mathbf{k}} = u_{\mathbf{k}} - \frac{1}{2} \langle |f_{\mathbf{k}}|^2 \rangle$.

For convenience, we call the model without the term $\sqrt{u} f(\mathbf{x}) \varphi^2(\mathbf{x})$ the pure model, since this term simulates the effect of random impurities [3].

Proof of the theorem. According to the rules of the diagram technique, the formal expansion for $G(\mathbf{k})$ involves all connected diagrams with two fixed outer solid lines. In the case of the pure model, diagrams are constructed of the vertices $\text{---}\text{---}\text{---}$, with factor $-uV^{-1}\tilde{u}_{\mathbf{k}}$ related to any zigzag line with wave vector \mathbf{k} . The solid lines are related to the correlation function in the Gaussian approximation $G_0(\mathbf{k}) = 1/(2r_0 + 2ck^2)$. Summation over the components $\varphi_i(\mathbf{k})$ of the vector $\varphi(\mathbf{k})$ yields factor n corresponding to each closed loop of solid lines in the diagrams. According to this, the formal perturbation expansion is defined at arbitrary n . In the limit $n \rightarrow 0$, all diagrams of $G(\mathbf{k})$ vanish except those which do not contain the closed loops. In such a way, for the pure model we obtain the expansion

$$G(\mathbf{k}) = \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \dots \quad (18)$$

In the case of the random model, the diagrams are constructed of the vertices $\text{---}\text{---}\text{---}$ and $\text{---}\text{---}\text{---}$. The factors $uV^{-1} \langle |f_{\mathbf{k}}|^2 \rangle$ correspond to the coupled dotted lines and the factors $-uV^{-1}u_{\mathbf{k}}$ correspond to the dashed lines. Thus, we have

$$G(\mathbf{k}) = \text{---}\text{---}\text{---} + \left[\text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} \right] + \dots \quad (19)$$

In the random model, first the correlation function $G(\mathbf{k})$ is calculated at a fixed $\{f_{\mathbf{k}}\}$ (which corresponds to connected diagrams where solid lines are coupled, but the dotted lines with factors $-\sqrt{u}V^{-1/2}f_{\mathbf{k}}$ are not coupled), performing the averaging with the weight $P(\{f_{\mathbf{k}}\})$ over the configurations of the random variable (i.e., the coupling of the dotted lines) afterwards. According to this procedure, the diagrams of the random model in general (not only at $n \rightarrow 0$) do not contain parts like $\text{---}\text{---}\text{---}$, $\text{---}\text{---}\text{---}$, $\text{---}\text{---}\text{---}$, etc., which would appear only if unconnected (i.e., consisting of separate parts) diagrams would be considered before the coupling of dotted lines.

It is evident from Eqs. (18) and (19) that all diagrams of the random model are obtained from those of the pure model if any of the zigzag lines is replaced either by a dashed or by a dotted line, performing summation over all such possibilities. Such a method is valid in the limit $n \rightarrow 0$, but not in general. The problem is that, except the case $n \rightarrow 0$, the diagrams of the pure model contain parts like $\text{---}\text{---}\text{---}$, $\text{---}\text{---}\text{---}$, $\text{---}\text{---}\text{---}$, etc. If all the depicted here zigzag lines are replaced by the dotted lines, then we obtain diagrams which are not allowed in the random model, as explained before. At $n \rightarrow 0$, the only problem is to determine the combinatorial factors for the diagrams obtained by the above replacements. For a diagram constructed of M_1 vertices $\text{---}\text{---}\text{---}$ and M_2 vertices $\text{---}\text{---}\text{---}$ the combinatorial factor is the number of possible different couplings of lines, corresponding to the given topological picture, divided by $M_1!M_2!$.

Our further consideration is valid also for the diagrams of free energy (at $n \rightarrow 0$ represented by the main terms containing single loop of solid lines) and of $2m$ -point correlation function. We define that all diagrams which can be obtained from the i -th diagram (i.e., the diagram of the i -th topology) of the pure model, belong to the i -th

group. Obviously, all diagrams of the i -th group represent a contribution of order u^l , where l is the total number of vertices $\text{---}\text{---}\text{---}$ in the i -th diagram. The sum of the diagrams of the i -th group can be found by the following algorithm.

1. Depict the i -th diagram of pure model in an a priori defined way.
2. Choose any one replacement of the vertices $\text{---}\text{---}\text{---}$ by $\text{---}\text{---}\text{---}$ and $\text{---}\text{---}\text{---}$, and perform the summation over all such possibilities. For any specific choice we consider only one of the equivalent $M_1!M_2!$ distributions of the numbered M_1 vertices $\text{---}\text{---}\text{---}$ and M_2 vertices $\text{---}\text{---}\text{---}$ over the fixed numbered positions instead of the summation over all these distributions with the weight $1/(M_1!M_2!)$. Thus, at this step the combinatorial factor for any specific diagram is determined as the number of possible distributions of lines (numbered before coupling) for one fixed location of vertices consistent with the picture defined in step 1.
3. The result of summation in step 2 is divided by the number of independent symmetry transformations (including the identical transformation) for the considered i -th diagram constructed of vertices $\text{---}\text{---}\text{---}$, since the same (original and transformed) diagrams were counted as different.

Note that the location of any vertex $\text{---}\text{---}\text{---}$ is defined by fixing the position of dashed line, the orientation of which is not fixed. According to this, the summation over all possible distributions of lines (numbered before coupling) for one fixed location of vertices yields factor $8^{M_1}4^{M_2/2}$. The i -th diagram of the pure model also can be calculated by such an algorithm. In this case we have 8^l line distributions, where $l = M_1 + M_2/2$ is the total number of vertices $\text{---}\text{---}\text{---}$ in the i -th diagram. Obviously, the summation of diagrams of the i -th group can be performed with factors 8^l instead of $8^{M_1}4^{M_2/2}$, but in this case twice smaller factors must be related to the coupled dotted lines. The summation over all possibilities where zigzag lines are replaced by dashed lines with factors $-uV^{-1}u_{\mathbf{k}}$ and by dotted lines with factors $\frac{1}{2}uV^{-1}\langle |f_{\mathbf{k}}|^2 \rangle$, obviously, yields a factor $uV^{-1}\left(-u_{\mathbf{k}} + \frac{1}{2}\langle |f_{\mathbf{k}}|^2 \rangle\right) \equiv -uV^{-1}\tilde{u}_{\mathbf{k}}$ corresponding to each zigzag line with wave vector \mathbf{k} . Thus, the sum over the diagrams of the i -th group is identical to the i -th diagram of the pure model defined by Eq. (17). By this the theorem has proved not only for the two-point correlation function, but also for $2m$ -point correlation function and free energy.


If, in general, the factor \sqrt{u} in Eq. (15) is replaced by $\sqrt{u'}$, where u' is an independent expansion parameter, then our analysis leads to the above relation between diagrams for $u\tilde{u}_{\mathbf{k}} = u u_{\mathbf{k}} - \frac{u'}{2}\langle |f_{\mathbf{k}}|^2 \rangle$. According to this, at $n \rightarrow 0$ the pure and random models cannot be distinguished within the diagrammatic perturbation theory. If, in principle, critical exponents can be determined from the diagram expansions at $n \rightarrow 0$, as it is suggested in the usual RG theory, then the same critical exponents should be provided for both models at $n \rightarrow 0$. In such a way, we conclude that the RG method is not correct because the above condition is violated. As compared to our simple treatment of the random model, the RG treatment includes additional Feynman diagrams because the Hamiltonian becomes more complicated after the renormalization. However, this does not enable to find the difference between both models: the original information, when one starts the perturbative renormalization of Hamiltonian (15), is contained in the Feynman diagrams we considered, but the renormalization by itself does not create new information

about the model. Really, by renormalization we merely “forget” some information about the short-wave fluctuations to make that for the long-wave fluctuations easier accessible. Thus, our conclusion remains true.

4 New method based on perturbation theory

As we have already discussed in Sect. 2, it is not a rigorous method to make a formal expansion like (12) and to try calculate the critical exponents therefrom. We propose another treatment of the diagrammatic perturbation theory. The basic idea is to obtain suitable equations by appropriate grouping of the diagrams. Suitable are such equations which allow to find the asymptotic expansions at the critical point directly in k power series, but not in terms of the formal parameter $\ln k$ (as in Eq. 12) which diverges at $k \rightarrow 0$. In such a way, for the Ginzburg–Landau model defined by Eq. (17), where $u \tilde{u}_{\mathbf{k}} = u_{\mathbf{k}}$, (for simplicity here we consider the case of the scalar order parameter, i. e. $n = 1$) we have obtained the Dyson equation

$$\frac{1}{2G(\mathbf{k})} = r_0 + ck^2 - \frac{\partial D(G)}{\partial G(\mathbf{k})} \quad (20)$$

where $D(G)$ denotes a quantity, the diagram expansion of which involves all the so-called skeleton diagrams (constructed of the fourth-order vertices $\text{---}\text{---}\text{---}\text{---}$ with factors $-V^{-1}u_{\mathbf{k}}$ related to the dashed lines) without outer lines. Skeleton diagram is defined as a connected diagram containing no parts like . In distinction to the usual (simple) perturbation theory, the true correlation function $G(\mathbf{k})$ is related to the coupled solid lines instead of the Gaussian correlation function $G_0(\mathbf{k})$. In such a way, quantity $D(G)$ have to be considered as a function of discrete variables $G(\mathbf{k})$ corresponding to the set of discrete wave vectors \mathbf{k} . Eq. (20) has been previously obtained in Ref. [5], where also the simplest (ring) skeleton diagrams have been considered. We have found a possibility to include all the skeleton diagrams into consideration, which allows to find the set of possible values for exact critical exponents. At $u \rightarrow 0$ our equations, represented by converging sums and integrals, define the true correlation function $G(\mathbf{k})$ with an error smaller than u^l at any positive l . This is quite enough to find the exact critical exponents based on general scaling properties of the solution in vicinity of the critical point. The structure of our equations allows to prove these properties taking into account all diagrams. The results also can be easily generalized to the case with $O(n)$ symmetry. It is not possible in this relatively short paper to give the mathematical derivations of our equations and results. Hopefully, they will be available in nearest future. Here we present the final result according to which possible exact values of critical exponents γ (susceptibility exponent) and ν (correlation length exponent) for the n -component vector model ($n = 1, 2, 3$, etc.) in $d < 4$ dimensions (i. e., at $d = 2, 3$) are given by

$$\gamma = \frac{d + 2j + 4m}{d(1 + m + j) - 2j}; \quad \nu = \frac{2(1 + m) + j}{d(1 + m + j) - 2j}, \quad (21)$$

where m may have a natural value starting with 1 and j is integer equal or larger than $-m$. In general, different values of j and m can correspond to different (natural) n , i. e., $j = j(n)$ and $m = m(n)$. It is easy to verify that at $j = 0$ and $m = 3$ Eq. (21) reproduces

the known [1] exact results in two dimensions. The known exact exponents for the spherical model [1] ($n = \infty$) are obtained at $j(n)/m(n) \rightarrow \infty$. Although the derivations are true for $d < 4$, Eq. (21) provides correct result $\nu = 1/2$ and $\gamma = 1$ also at $d = 4$. It is reasonable to consider d as a continuous parameter. This leads to the conclusion that $m = 3$ and $j = 0$ are the correct values for the case $n = 1$ not only at $d = 2$, but also at $d = 3$. In the latter case we have $\gamma = 5/4$ and $\nu = 2/3$. The nearest values of γ and ν provided by Eq. (21), e. g., at $j = 1$ and $m = 3$ or at $j = 1$ and $m = 4$ are then the most probable candidates for the case $n = 2$.

5 Comparison of results and discussion

It is commonly believed that all more or less correct Monte Carlo (MC) simulations confirm the values of critical exponents obtained from the perturbation expansions based on the renormalization group. This is not true. We have found that some kind of MC simulations at the critical point, namely, the MC simulations of fractal configurations of Ising model [6] and the MC simulations of the energy density [7] for the XY model in reality do not confirm the results of the RG theory, but provide the values of critical exponents which are very close to those we predicted.

The MC simulations of Ref. [6] allows to determine the fractal dimensionality D (the largest cluster in the relevant configuration has the volume L^D where L denotes the linear size of the system) which is related to the critical exponents by $\gamma = \nu(2D - d)$ or, which is the same, $\eta = 2 - \gamma/\nu = d + 2 - 2D$. In our opinion, this method is better than other more convenient simulation methods, since it provides the value of η as a result of direct simulation, i. e., there are no fitting parameters. Besides, the result is relatively insensitive to the precise value of the critical coupling (temperature). In Fig. 1 we have shown the average values of D (the averaging is made over the MC steps from 1 to 10 (except the initial point), from 11 to 20, and so on) calculated from the MC data of Ref. [6] by measuring deviation from the line $D = 2.48$ in Fig. 8 (of Ref. [6]). If properly treated, these simulation data confirm the value of η about $1/8$ (or $D = 2.4375$) consistent with our prediction $\gamma = 5/4$ and $\nu = 2/3$, as it is evident from Fig. 1. The value $D = 2.46 \pm 0.01$ reported in Ref. [6] seems to be determined from the upper MC points (Fig. 8 in Ref. [6]) only which are closer to the known theoretical prediction $D = 2.48$.

As regards the MC simulations of the energy density E of XY model [7] at the critical point, the true picture can be reconstructed from the simulated values listed in Tab. I of Ref. [7]. Since all the values of E are of comparable accuracy, it is purposeful to use the least-square method to find the optimum value of $1/\nu$ by fitting the MC data to the prediction of the finite-size scaling theory

$$E(L) = E_0 + E_1 L^{\frac{1}{\nu} - d}, \quad (22)$$

where $E(L)$ is the energy density at the critical temperature T_λ depending on the linear size of the system L . The standard deviation of the simulated data points from the analytical curve (22) can be easily calculated for any given value of $1/\nu$ with the parameters E_0 and E_1 corresponding to the least-square fit. The result is shown in Fig. 2. The thick solid curve is calculated including all 11 data points ($L=10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 80$), whereas the dashed line – including 9 data points (except $L=10, 15$) used

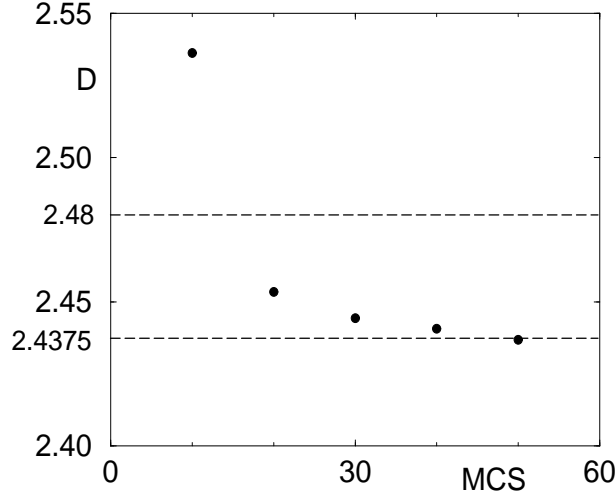


Figure 1: Fractal dimension D of the three dimensional Ising model at the critical point simulated by Monte Carlo method (MCS means Monte Carlo steps). The upper and lower dashed lines indicate the theoretical values expected from the known and from our critical exponents, respectively.

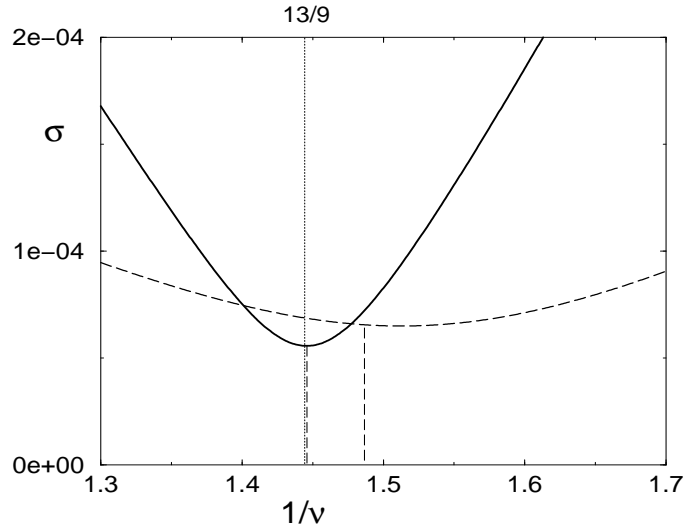


Figure 2: The standard deviation σ vs the value of $1/\nu$ used in the least-square fit of the finite-size scaling curve to the simulated results including 11 data points (solid curve) and 9 data points (dashed curve). Minimum of the solid curve, shown by a vertical dashed line, corresponds to the best fit $1/\nu = 1.4457$ which is close to our theoretical value $13/9$ indicated by a vertical dotted line. Other vertical dashed line indicates the value 1.487 proposed by authors of Ref. [7].

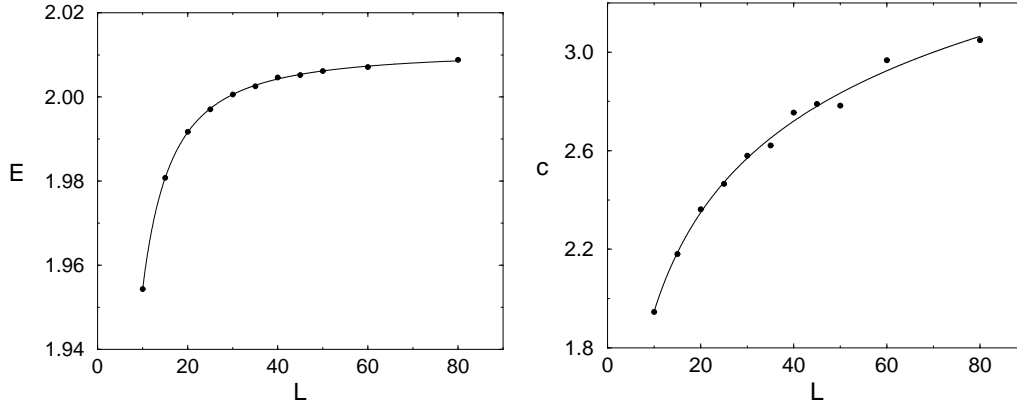


Figure 3: Our fits to the original MC data of Ref. [7] for the energy density (left) and for the specific heat (right) depending on the linear size of the system L .

for the fitting in Ref. [7]. Minimum of the solid curve, shown by a vertical dashed line, corresponds to the best fit $1/\nu = 1.4457$ which comes very close to our theoretical value $13/9$ (provided by (21) at $j = 1$ and $m = 3$) indicated by a vertical dotted line. We have estimated the statistical error of this MC result about ± 0.007 by comparing the best fits for several random data sets. Different data sets have been generated from the original one by omitting some data points with $10 < L < 80$. We have found it unreasonable to omit the data points with two smaller sizes, as it has been proposed in Ref. [7], since the result in this case becomes very poorly defined, i. e., the dashed curve in Fig. 2 has a very broad minimum. Besides, there is no reason to omit the smallest sizes, since the analytical curve (22) excellently fit all the data points and the standard deviation for 11 data points is even smaller than that for 9 data points (see Fig. 2). The possible systematical error due to the inaccuracy in the critical temperature $T_\lambda = 2.2017 \pm 0.0005$ (the error bars are taken from the source of this estimation [8]) used in the simulations [7] has been evaluated ± 0.017 by comparing the simulation results at T_λ values 2.2012, 2.2017, and 2.2022. In this case the values of the energy density at a slightly shifted temperature have been calculated from the specific heat data given in Tab. I of Ref. [7]. In such a way, our final estimate from the original MC data of Ref. [7] is $1/\nu = 1.446 \pm 0.025$ in a good agreement with our theoretical value $13/9 = 1.444\dots$ and in a clear disagreement with the usual (RG) prediction about 1.493. One can only wonder where the value 1.487 proposed in Ref. [7] comes from. It does not correspond neither to the best fit for 11 data points nor to that for 9 data points, as it is evident from Fig. 2. The values of $1/\nu$ and α/ν cannot be determined independently from the discussed here energy density data. One of them have to be calculated from the scaling relation $\alpha/\nu + d = 2/\nu$. If authors of Ref. [7] were able to determine $1/\nu$ with ± 0.081 accuracy, then they should be able to find α/ν with ± 0.162 accuracy. In this aspect, the estimate $\alpha/\nu = -0.0258 \pm 0.0075$ given by the authors looks more than strange.

In Fig. 3 we have shown our fits to the MC data for the energy density $E(L) = 2.0108 - 2.0286 L^{-14/9}$ and for the specific heat $c(L) = 7.360 - 6.990 L^{-1/9}$. They do not look worse than those in Ref. [7], but our fit for $c(L)$ seems to be better.

One believes that the value of critical exponent ν about 0.67, predicted by the RG

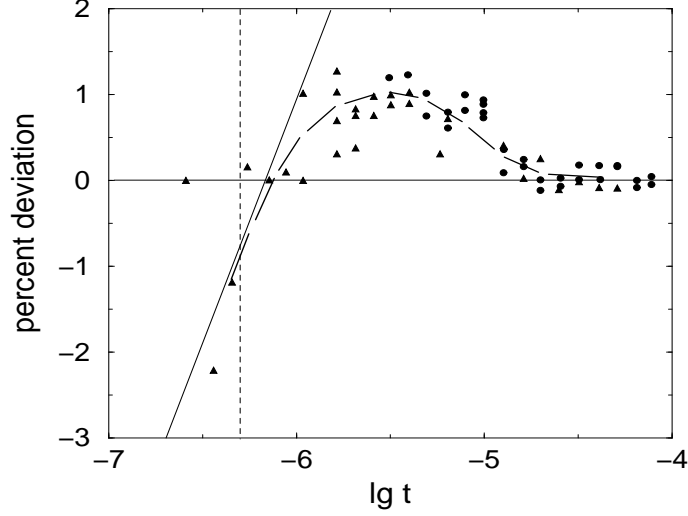


Figure 4: Percent deviation of the experimental ρ_s/ρ data [9] from the expected theoretical relation (Eq. (23) at $\delta = 0$). The straight line shows the slope of this plot at the value of t , equal to $5 \cdot 10^{-7}$, indicated by a vertical dashed line.

theory at $n = 2$, is well confirmed by very accurate measurements of the superfluid fraction $\rho_s/\rho = y$ in ^4He . This is not true, since in reality these experiments [9] provide a good evidence that the effective critical exponent $\nu_{eff}(t) = \partial(\ln y)/\partial(\ln t)$ remarkably increases when the reduced temperature $t = (T_\lambda - T)/T_\lambda$ (where T_λ is the critical temperature) is decreased below 10^{-5} . According to Ref. [9], ρ_s/ρ is given by

$$\rho_s/\rho = y(t) = k_0(1 + k_1 t)(1 + D_\rho t^\Delta)t^\zeta \times (1 + \delta(t)) , \quad (23)$$

where k_0 , k_1 , D_ρ , and ζ are the fitting parameters, $\Delta = 0.5$ is supposed to be the correction-to scaling exponent, and $\delta(t)$ is the measured relative deviation from the expected theoretical expression obtained by setting $\delta(t) = 0$. The percent deviation discussed in Ref. [9] is 100 times $\delta(t)$. From Eq. (23) we obtain

$$\nu_{eff}(t) = \zeta + \frac{k_1 t}{1 + k_1 t} + \frac{\Delta D_\rho t^\Delta}{1 + D_\rho t^\Delta} + \frac{1}{1 + \delta(t)} \times \frac{\partial \delta(t)}{\partial(\ln t)} . \quad (24)$$

For the values of t as small as $t < 10^{-5}$ and for $\delta(t) \ll 1$ Eq. (24) with the fitting parameters $\zeta = 0.6705$, $k_0 = 2.38$, $k_1 = -1.74$, and $D_\rho = 0.396$ used in Ref. [9] reduces to

$$\nu_{eff}(t) \simeq \zeta + \partial \delta(t)/\partial(\ln t) . \quad (25)$$

The second term in this equation is proportional to the slope of the percent deviation plot $100 \delta(t)$ vs $\ln t$ or $\lg t$ (the decimal logarithm) in Figs. 2 and 3 of Ref. [9]. We have read the experimental data from Fig. 2 in Ref. [9] within the region $t < 10^{-4}$ and have depicted them in Fig. 4. Almost all the data points with a reasonable accuracy fit the smooth curve $\delta(t)$ vs $\lg t$ (dashed line) having a maximum at about $\lg t = -5.5$. It means that $\partial^2 \delta(t)/\partial(\ln t)^2$ is negative within some region around the maximum, i. e., according

to (25) the effective critical exponent $\nu_{eff}(t)$ increases if t is decreased. We have roughly estimated and have shown by straight line the slope of this curve at $t = t^* = 5 \cdot 10^{-7}$ (t^* value is indicated in Fig. 4 by vertical dashed line). From this we obtain $\partial\delta(t)/\partial(\ln t) \approx 0.025$. This result depends on the shift in the experimentally determined T_λ value. To obtain a more reliable estimate, we have performed the same manipulations with the data depicted in Fig. 3 of Ref. [9] corresponding to T_λ shifted by $\pm 20nK$, and have obtained the values of $\partial\delta(t)/\partial(\ln t)$ about 0.03 and 0.015, respectively. Our final result 0.0233 ± 0.0083 for this derivative at $t = t^*$ has been obtained by averaging over the three above discussed estimates (0.015, 0.025, and 0.03) with the error bars large enough to include all these values. According to this, from Eq. (25) with $\zeta = 0.6705$ we obtain $\nu_{eff}(t^*) = 0.694 \pm 0.009$ which, again, is in a good agreement with the value $\nu = 9/13 \simeq 0.6923$ provided by Eq. (21) at $j = 1$ and $m = 3$ and in a disagreement with the RG predictions.

6 Conclusions

We have proposed a novel method (Sect. 4) which allows to predict the exact values of critical exponents in the Ginzburg–Landau phase transition model. Our proposal is accompanied by a critical analysis of the conventional (perturbative) RG method. In view of this analysis (Sect. 2 and 3) and comparison with MC simulation results and experiments (Sect. 5), our results should not be doubted from the positions of the conventional (RG) theory. The best evidence of the correctness of our treatment is the precise agreement with the known exact solutions.

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